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Optimal Convergence of Basic Schemes for Elliptic Boundary Value Problems with Strong Parabolic Layers

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We consider a convection-diffusion problem with strong parabolic boundary layers and its discretization using upwind finite differences or bilinear finite elements on a layer-adapted mesh. Based on a new decomposition of the solution we are able to prove optimal uniform convergence results.

1. INTRODUCTION

In recent years it has become quite popular to use piecewise uniform grids to handle boundary and interior layers; for a survey up to 1998 see [11]. Today the case of exponential layers is particularly well understood, but for the case of parabolic layers, which are in practice more important, the situation is much less clear. Consider the model problem

$$\begin{aligned} L_\varepsilon u := -\varepsilon\Delta u + u_x &= f \qquad \text{in } \Omega = (0,1)^2, \\ u &= 0 \qquad \text{on } \Gamma = \partial\Omega, \end{aligned} \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small parameter. The problem is characterized by an exponential layer at $x = 1$ and parabolic layers (see Section 2 for a detailed discussion) at $y = 0$ and $y = 1$. Then it is well known that there does not exist a fitted scheme that converges uniformly on a uniform mesh [15]. As a consequence layer-adapted meshes are used. Let the number of subintervals for the discretization in both the x and y directions be N. In the book [14] one finds, for the upwind finite difference scheme on a

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Shishkin mesh, uniform convergence results of the form

$$\|u - u^N\|_{\infty, d} \leq C(N^{-1} \ln^2 N)^p, \tag{2}$$

with $p = 1/14$ or $p = 1/18$. Here $\|\cdot\|_{\infty, d}$ denotes the discrete maximum norm and C (here and throughout the paper) is a generic constant independent of ε and N.

Numerical experiments show, however, a rate significantly better than the existing theory predicts [5]. Even for problems with weaker layers, replacing the Dirichlet boundary conditions with boundary conditions of the third kind [3], the theoretical rate proved (1/3) is less than the numerical rate observed for problems with strong layers.

Based on a new decomposition of the solution presented in Section 2 we are able to prove the following:

If f is smooth and satisfies the compatibility condition $f(P_i) = 0$ at the four corners P_i of $\overline{\Omega}$, then the error for simple upwinding on a Shishkin mesh satisfies

$$\|u - u^N\|_{\infty, d} \leq CN^{-1} \ln N. \tag{3}$$

What about finite elements? For finite elements it is not necessary to have classical solutions, and in fact we present a partially new error analysis *without the compatibility condition* $f(P_i) = 0$. Because we believe this analysis is also interesting for problems with exponential layers, we first present the technique for that case in Section 3. Finally, we consider in Section 4 finite elements for Problem (1) and prove for the error in the $\varepsilon^{1/2}$-weighted H^1-seminorm the following:

If f is smooth, then the finite-element approximation of Problem (1) *using bilinear or linear elements on a Shishkin mesh satisfies*

$$\varepsilon^{1/2} |u - u^N|_1 \leq CN^{-1} \ln N.$$

For problems with strong parabolic boundary layers this is, to our knowledge, the first result of its kind.

2. ASYMPTOTIC EXPANSIONS AND MODIFIED DECOMPOSITIONS

Let us assume that the function f is sufficiently smooth (for instance, it is sufficient to take $f \in C^{3,\alpha}(\overline{\Omega})$) and satisfies the compatibility condition $f(P_i) = 0$ at the four corners of $\overline{\Omega}$. Then $u \in C^{3,\alpha}(\overline{\Omega})$ [4].

Note that it makes no sense to require more smoothness of u because the corresponding compatibility conditions then are no longer a *local* functional if the given problem has nonconstant coefficients.

The analysis of the upwind finite-difference scheme on a Shishkin mesh can be based on the analysis of the consistency error and the application of suitably chosen barrier functions; see for instance [8]. In [8] a problem with exponential layers is studied, but if we have the necessary a priori information there is no difficulty in applying the technique to a problem with characteristic boundaries.

For the analysis of the consistency error we have to estimate integrals containing third-order derivatives. Therefore, it would be desirable to have a decomposition of the solution of (1) of the form:

$$u = S + E_1 + E_2 + E_3. \tag{4a}$$

Here S and E_1, E_2, E_3 should satisfy the following estimates for $i = 0$, $0 \le j \le 3$, and $j = 0$, $0 \le i \le 3$.

$$\left| \frac{\partial^{i+j} S}{\partial x^i \partial y^j} \right| \le C \tag{4b}$$

$$\left| \frac{\partial^{i+j} E_1}{\partial x^i \partial y^j} \right| \le C\varepsilon^{-i} e^{-(1-x)/\varepsilon} \tag{4c}$$

$$\left| \frac{\partial^{i+j} E_2}{\partial x^i \partial y^j} \right| \le C\varepsilon^{-j/2} B(y) \tag{4d}$$

$$\left| \frac{\partial^{i+j} E_3}{\partial x^i \partial y^j} \right| \le C\varepsilon^{-(i+j/2)} e^{-(1-x)/\varepsilon} B(y) \tag{4e}$$

The function B is defined by $B(y) = \exp(-\gamma y/\sqrt{\varepsilon}) + \exp(-\gamma(1-y)/\sqrt{\varepsilon})$ for some arbitrary constant $\gamma > 0$. The term E_1 reflects the exponential layer at $x = 1$; E_2 majorizes the parabolic layers at $y = 0$ and $y = 1$, and E_3 majorizes the corner layers at $(1, 0)$ and $(1, 1)$.

We shall show that our assumptions on f are sufficient for the existence of the decomposition (4). We remark that in [14, Chapter IV, pp. 140–143], a solution decomposition is formulated for a problem much more general than (1), but neither a detailed proof that this decomposition exists is given nor is it easy to see that for a smooth f with $f(P_i) = 0$ the decomposition simplifies to (4) for the model problem (1).

Before we describe the possible definitions of S, E_1, E_2, and E_3 and prove the above estimates, we first discuss two approaches from the literature.

Higher Order Asymptotic Expansions

Il'in and Lelikova succeeded in constructing higher-order asymptotic expansions for the solution of problem (1) as early as 1975—see Il'in's book [6, Chap. IV, Sect. 1]. Later, Shih reproduced these results in his dissertation [13].

Let us consider the parabolic boundary layer at $y = 0$, for instance. Setting $\zeta = y/\sqrt{\varepsilon}$, parabolic layer functions z_k are usually defined as the solutions of

$$\begin{aligned} -\frac{\partial^2 z_k}{\partial \zeta^2} + \frac{\partial z_k}{\partial x} &= \begin{cases} 0 & k = 0; \\ \dfrac{\partial^2 z_{k-1}}{\partial x^2} & k \neq 0; \end{cases} \qquad z_k(0, \xi) = 0, \\ z_k(x, 0) &= \begin{cases} -u_0(x, 0) & k = 0, \\ 0 & k \neq 0. \end{cases} \end{aligned} \tag{5}$$

Here u_0 is the smooth solution of the reduced problem $(u_0)_x = f$, $u_0|_{x=0} = 0$. To bound the derivative of order $\ell + 2$ of z_0 with respect to x, we need the compatibility condition

$$\frac{\partial^{\ell+1}}{\partial x^{\ell+1}} u_0(0, 0) = 0 \qquad (\ell \geq 0). \tag{6}$$

Our assumption $f(0, 0) = 0$ guarantees the boundedness of $(z_0)_{xx}$. If we wish to construct an expansion with, for instance, a remainder of order $0(\varepsilon^2)$ we need more smoothness, but the necessary compatibility conditions are not automatically satisfied. Therefore, Il'in and Lelikova modify the definition of z_k as follows:

$$\text{Replace} \quad u_0(x, 0) \quad \text{with} \quad u_0(x, 0) - \sum_{i=1}^{M} \frac{x^i}{i!} \frac{\partial^i}{\partial x^i} u_0(0, 0). \tag{7}$$

Then choosing M adequately, the necessary compatibility conditions are satisfied.

To compensate for the replacement (7), we introduce a new term in the asymptotic expansion, an *elliptic boundary layer* along the characteristic boundaries. With $\xi = x/\varepsilon$ and $\eta = y/\varepsilon$, the elliptic boundary-layer function v_k satisfies

$$-\left(\frac{\partial^2 v_k}{\partial \xi^2} + \frac{\partial^2 v_k}{\partial \eta^2} \right) + \frac{\partial v_k}{\partial \xi} = 0, \qquad v_k(0, \eta) = 0, \; v_k(\xi, 0) = w_k(\xi). \tag{8}$$

The functions w_k are polynomials in ξ corresponding to the terms added to u_0 in (7). In [6] the estimate

$$|v_k| \leq C e^{-c(\sqrt{\xi^2 + \eta^2} - \xi)} \tag{9}$$

is proved and further information on the derivatives of v_k can be found.

A mesh construction based on the information of the elliptic boundary-layer functions leads to a mesh at the characteristic boundaries that is much finer than the proposal of Shishkin [14] and contradicts the existing numerical experiments. *We feel that the standard technique of matched asymptotic expansions suffers from the approximation of smooth functions (here $C^{3,\alpha}(\overline{\Omega})$) by less smooth terms that arise artificially in the approximation of problems with strong parabolic layers.*

Therefore, modifications of standard asymptotic expansions are necessary.

A Decomposition by Karepova

Karepova [7] analyzes a special difference scheme using exponential fitting in the x-direction and a special layer-adapted mesh in the y-direction. She also assumes the compatibility condition $f(P_i) = 0$ at the four corners of $\overline{\Omega}$ and consequently has $u \in C^{3,\alpha}(\overline{\Omega})$.

For the analysis of the scheme, the new solution decomposition

$$u = U_0 + \varrho_0 + \varepsilon\eta, \tag{10}$$

where ϱ_0 corresponds to the exponential layer at $x = 1$, is used, while U_0 is the solution of the parabolic problem

$$-\varepsilon(U_0)_{yy} + (U_0)_x = f, \qquad U_0|_{y=0} = U_0|_{y=1} = 0, \qquad U_0|_{x=0} = 0. \tag{11}$$

Karepova states the following estimates for u, U_0, and η.

PROPOSITION.

(i) $\left|\dfrac{\partial^j u}{\partial y^j}\right| \le C(1 + \varepsilon^{-j/2}B(y)) \qquad \text{for} \quad j = 1, 2, 3$

(ii) $\left|\dfrac{\partial^k U_0}{\partial x^k}\right| \le C \qquad \text{for} \quad k = 1, 2,$

(iii) $\|\eta\|_\infty \le C, \qquad \left|\dfrac{\partial \eta}{\partial x}\right| \le C(1 + \varepsilon^{-1}\exp(-(1-x)/\varepsilon)).$

The derivation of the above estimates is based on the maximum principle, majorizing functions and differentiation of the given problem (1) to get additional information on the boundary. For first-order derivatives, this technique is well known. What about the higher-order derivatives in (i) and (ii)?

Let us sketch some details. The functions u_y, u_{yy}, and u_{yyy} satisfy the same differential equation as u. But how do we get information on the boundary values of u_{yy}, u_{yyy} at $y = 0$ and $y = 1$? Note that we do not necessarily need the functions themselves at the boundaries; other boundary conditions are possible.

The differential equation yields $|u_{yy}| \le C/\varepsilon$, then we get (i) for $j = 2$ using barrier functions. If we differentiate the equation twice, we get formally $|u_{yyyy}| \le C\varepsilon^{-2}$. But this differentiation and restriction to the boundary is unjustified because we have only $u \in C^{3,\alpha}(\overline{\Omega})$, so the proof of (i) for $j = 3$ is not rigorous.

To prove (ii) for $k = 2$ with barrier functions we need some information about $\partial^2 U_0/\partial x^2$ on $x = 0$. Once again we could get this information by formally differentiating the equation twice, but we need more compatibility to justify such a step.

Karepova's decomposition is a first step toward the avoidance of classical asymptotic expansions, but her proof of the proposition above is not rigorous. Furthermore, her decomposition yields only limited information on the remainder η, and it is not clear that this information is sufficient to analyze a difference scheme.

Elliptic Decomposition

Let us define a decomposition of the solution of problem (1) as

$$u = S + E_1 + E_2 + E_3,$$

in such a way that S, E_1, E_2, and E_3 are the solutions of some *elliptic* boundary value problems. The smooth part S solves

$$\begin{aligned} L_\varepsilon S &= f^* \quad \text{in the half-plane } x > 0 \\ S &= 0 \qquad \text{on } x = 0. \end{aligned} \tag{12a}$$

Here and in what follows, given a function g defined on $\overline{\Omega}$, we use g^* to denote some extension of g that has compact support.

Because S is defined as the solution of an elliptic problem in an unbounded domain, we need some growth condition to yield uniqueness and the validity of a comparison principle. We have from [10, Chap. 2, Sect. 9] the following.

Phragmén–Lindelöff Principle. Let Ω be a unbounded domain, and let u satisfy

$$\begin{aligned} Lu &\le 0 \qquad \text{in } \Omega, \\ u &\le 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Suppose there is a function w with the properties

$$\begin{aligned} Lw &\ge 0 \qquad \text{in } \Omega, \\ w &> 0 \qquad \text{on } \Omega \cup \partial\Omega, \\ \lim_{|x|\to\infty,\, x\in\Omega} w &= \infty. \end{aligned}$$

Then the growth condition

$$\lim_{A\to\infty} \inf \left[\sup_{w(x)=A,\, x\in\Omega} \frac{u(x)}{w(x)} \right] \le 0$$

implies $u \le 0$ in Ω.

The Phragmén–Lindelöff Principle applied to $u_1 - u_2$ also yields a comparison principle, assuming that both of the functions u_1 and u_2 satisfy the growth condition. As a special case, we have uniqueness for the solution of the boundary value problem in the class of functions satisfying the growth condition.

Consider the problem (12a). The function w defined by

$$w(x) = 1 + \varepsilon x + x^2/2$$

satisfies the condition of the Phragmén–Lindelöff Principle. Therefore, we work in the class of functions that do not grow as rapidly as w and obtain uniqueness and the validity of a comparison principle. For instance, the barrier function Cx majorizes S and yields additionally that $|S_x|_{x=0}| \le C$.

For the following boundary value problems, when defining E_1 and E_2 similar arguments hold true, so we omit the corresponding growth condition.

As outlined above, we next define

$$\begin{aligned} L_\varepsilon E_1 &= 0 \text{ in the strip} \quad 0 < x < 1, \\ E_1 &= 0 \text{ for } x = 0; \quad E_1 = -S^* \quad \text{for } x = 1. \end{aligned} \tag{12b}$$

It is intuitively clear that E_1 satisfies (4c); Appendix C of [14] contains a full discussion of a more general problem in a strip.

Next, we define E_2 by

$$\begin{aligned} L_\varepsilon E_2 &= 0 \quad \text{in } x > 0,\ 0 < y < 1, \\ E_2 &= 0 \text{ on } x = 0; \quad E_2 = -S^* \text{ for } y = 0 \quad \text{and} \quad y = 1. \end{aligned} \tag{12c}$$

Note that the Dirichlet boundary conditions are compatible at the corners $(0, 0)$ and $(0, 1)$.

We use the maximum principle and barrier functions to estimate E_2 and its first-order derivatives. To bound $(E_2)_{xx}$ and $(E_2)_{xxx}$ we need the information that these quantities are bounded at $x = 0$. Setting

$$E_2 = -S + u_0 + \widehat{E}_2,$$

where u_0 is the (smooth) solution of the reduced problem, $\widehat{E}_2$ is the solution of

$$\begin{aligned} L_\varepsilon \widehat{E}_2 &= \varepsilon \triangle u_0 \quad \text{in } x > 0,\ 0 < y < 1, \\ \widehat{E}_2 &= 0 \text{ on } x = 0; \quad \widehat{E}_2 = 0 \quad \text{for } y = 0 \text{ and } y = 1. \end{aligned} \tag{12d}$$

Restricting the differential equation to $x = 0$, the boundedness of $(\widehat{E}_2)_x|_{x=0}$ implies the boundedness of $(\widehat{E}_2)_{xx}|_{x=0}$ and thus of $(E_2)_x|_{x=0}$. With similar arguments one can prove that the third-order derivative is bounded at $x = 0$.

To estimate the derivatives of E_2 with respect to y, we first bound the derivatives at $y = 0$ and $y = 1$. Restricting E_2 to a finite rectangle containing $\overline{\Omega}$, we use the technique of [9] to obtain preliminary bounds on the derivatives of E_2,

$$\left|\frac{\partial^j E_2}{\partial y^j}\right| \le C\varepsilon^{-j/2} \qquad \text{for } j = 1, 2, 3.$$

With these bounds on $y = 0$ and $y = 1$, barrier functions of the type

$$C \exp(\sigma x) B(y),$$

with a suitably chosen parameter σ, which were used in [7], result in

$$\left|\frac{\partial^j E_2}{\partial y^j}\right| \le C\varepsilon^{-j/2} B(y) \qquad \text{for } j = 1, 2, 3.$$

Finally, we define E_3 by

$$\begin{aligned} L_\varepsilon E_3 &= 0 \qquad \text{in } \Omega, \\ E_3 &= -E_2 \quad \text{on } x = 1, \qquad E_3 = 0 \quad \text{on } x = 0, \\ E_3 &= -E_1 \quad \text{on } y = 0 \quad \text{and} \quad y = 1. \end{aligned} \tag{12e}$$

We further decompose $E_3 = E_3^{(1)} + E_3^{(2)}$, where both $E_3^{(1)}$ and $E_3^{(2)}$ satisfy the homogeneous equation and the homogeneous condition at $x = 0$, but

$$E_3^{(1)}|_{y=1} = 0, \qquad E_3^{(1)}|_{y=0} = -E_1, \qquad E_3^{(1)}|_{x=1} = -(1-y)E_2$$

and

$$E_3^{(2)}|_{y=1} = -E_1, \qquad E_3^{(2)}|_{y=0} = 0, \qquad E_3^{(2)}|_{x=1} = -yE_2.$$

Considering, for instance, $E_3^{(1)}$, the transformation

$$E_3^{(1)} = e^{-\frac{1-x}{\varepsilon}} e^{-\gamma y/\sqrt{\varepsilon}} \widehat{E}_3^{(1)}$$

results in an elliptic problem for $\widehat{E}_3^{(1)}$ with a structure similar to (1) with bounded Dirichlet boundary conditions. The technique of [9] yields

$$\left|\frac{\partial^{i+j} \widehat{E}_3^{(1)}}{\partial x^i \partial y^j}\right| \le C\varepsilon^{-(i+j/2)} \qquad \text{for } i + j \le 3.$$

Then for E_3 itself we get Eq. (4e).

Summarizing, we have proved the following:

THEOREM 1. *If* $f \in C^{3,\alpha}(\overline{\Omega})$ *and satisfies the compatibility condition* $f(P_i) = 0$ *at the four corners* P_i *of* $\overline{\Omega}$, *then the solution of the boundary value problem* (1) *satisfies a decomposition of the form* (4a), *where the derivatives up to order three with respect to x or y satisfy Eqs*. (4b), (4c), (4d), *and* (4e).

Combining Theorem 1 with the well-known technique used to analyze the upwind finite-difference scheme [8], we obtain the uniform convergence result stated in the Introduction.

3. FINITE ELEMENTS FOR A PROBLEM WITH EXPONENTIAL LAYERS

Let us consider instead of (1) the problem with exponential layers,

$$\begin{aligned} L_\varepsilon u := -\varepsilon \Delta u + u_x + u_y = f \qquad & \text{in } \Omega = (0,1)^2 \\ u = 0 \qquad & \text{on } \Gamma = \partial\Omega. \end{aligned} \tag{13}$$

Now we assume that only $f \in C^{1,\alpha}(\overline{\Omega})$ *and do not require the compatibility condition* $f(P_i) = 0$ *at the corners of* $\overline{\Omega}$.

Then the solution of problem (13) with exponential layers at $x = 1$ and $y = 1$ can be decomposed as

$$u = S + E_1 + E_2 + E_3, \tag{14}$$

with

$$\begin{aligned} \left| \frac{\partial^{i+j}}{\partial x^i y^j} S \right| &\leq C; \left| \frac{\partial^{i+j}}{\partial x^i y^j} E_1 \right| \leq C\varepsilon^{-i} \varepsilon^{-(1-x)/\varepsilon}, \\ \left| \frac{\partial^{i+j}}{\partial x^i y^j} E_2 \right| &\leq C\varepsilon^{-j} \varepsilon^{-(1-y)/\varepsilon}, \end{aligned} \tag{15}$$

and

$$\left| \frac{\partial^{i+j}}{\partial x^i y^j} E_3 \right| \leq C\varepsilon^{-(i+j)} e^{-(1-x)/\varepsilon} e^{-(1-y)/\varepsilon} \qquad \text{for } i + j \leq 1.$$

Here S, E_1, E_2, E_3 are defined by:

$$L_\varepsilon S = f^* \text{ in } x > 0, y > 0; \qquad S|_{x=0} = 0,\ S|_{y=0} = 0, \tag{16a}$$

$$\begin{aligned} L_\varepsilon E_1 = 0 \text{ in } 0 < x < 1; \qquad & y > 0; \\ E_1|_{y=0} = 0,\ E_1|_{x=0} = 0, E_1|_{x_1=1} &= -S^* \end{aligned} \tag{16b}$$

$$L_\varepsilon E_2 = 0 \text{ in } 0 < y < 1;\ x > 0;$$

$$E_2|_{y=0} = 0,\ E_2|_{x=0} = 0,\ E_2|_{y=1} = -S^* \tag{16c}$$

$$L_\varepsilon E_3 = 0 \text{ in } \Omega,\ E_3|_{x=0} = E_3|_{y=0} = 0;$$

$$E_3|_{x=1} = -E_2,\ E_3|_{y=1} = -E_1. \tag{16d}$$

Now let us consider bilinear (or linear) finite elements on a Shishkin mesh. It is clear that the use of only (15) for the first-order derivatives does not allow us to prove optimal-order robust error estimates. We need some further information on second-order derivatives and shall see that it is sufficient to have some rough estimates in Sobolev norms.

In the following we combine the techniques of [16] and [2] with some new ideas to avoid the use of the pointwise estimates (15) for second-order derivatives.

Let us follow [2] and denote by Ω_{22} the part of Ω where all layer functions are already small, i.e., are of order $O(N^{-2})$. By Ω_{11} we denote the region around $(1, 1)$ with fine mesh sizes in both the x and y directions. Ω_{12} and Ω_{21} are characterized by anisotropic elements; we set $\Omega_E := \Omega_{11} \cup \Omega_{12} \cup \Omega_{21}$. See Fig. 1.

Let us repeat the basic idea of the finite element analysis of [16]. Denoting the bilinear form related to the weak formulation of problem (13) by $a(\cdot, \cdot)$, we have

$$\begin{aligned}
\varepsilon|u^I - u^N|^2 &= a(u^I - u^N, u^I - u^N)\\
&= a(u^I - u, u^I - u^N) + a(u - u^N, u^I - u^N)\\
&= a(u^I - u, u^I - u^N)\\
&= \varepsilon(\nabla(u - u^I), \nabla(u^I - u^N)) + (u - u^I, \nabla(u^I - u^N)).
\end{aligned}$$

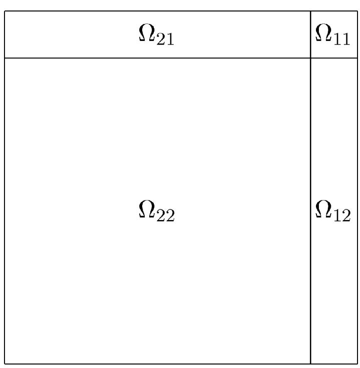

FIG. 1. S-mesh for a problem with exponential boundary layers.

The first term is simply estimated by Cauchy–Schwarz, while for the second term different techniques are used.

LEMMA 1. *If the interpolation error satisfies the estimates*

$$\varepsilon^{1/2}|u-u^I|_1 \le CN^{-1}\ln N;$$
$$\|u-u^I\|_{0,\Omega_{22}} \le CN^{-2}; \qquad \|u-u^I\|_{0,\Omega_E} \le C\varepsilon^{1/2}(N^{-1}\ln N), \tag{17}$$

then we obtain for the discretization error

$$\varepsilon^{1/2}|u-u^N|_1 \le CN^{-1}\ln N.$$

As sketched above, the essential ingredient in proving the lemma is the estimate for the convection term. On Ω_{22} an inverse estimate is applied:

$$|(u-u^I, \nabla(u^I-u^N))_{\Omega_{22}}| \le \|u-u^I\|_{0,\Omega_{22}} N\|u^I-u^N\|_{0,\Omega_{22}}.$$

On the other hand, on Ω_E we have

$$|(u-u^I, \nabla(u^I-u^N))_{\Omega_E}| \le \varepsilon^{-1/2}\|u-u^I\|_{0,\Omega_E}\varepsilon^{1/2}|u^I-u^N|_{1,\Omega_E}.$$

Thus Lemma 1 is proved.

Now we discuss the validity of the three interpolation error estimates used in Lemma 1.

For the layer part the second estimate of (17) follows from the definition of Ω_{22}, while for the smooth part we use

$$\|S-S^I\|_{0,\Omega_{22}} \le CN^{-2}\|S\|_{W^{2,1}}.$$

Furthermore, the a priori estimate from [1] applied to ∇S yields $\|S\|_{W^{2,1}} \le C$. Next,

$$\|u-u^I\|_{0,\Omega_E} \le (\text{meas}\ \Omega_E)^{1/2}\|u-u^I\|_{\infty,\Omega_E} \le C\varepsilon^{1/2}\|u-u^I\|_{\infty,\Omega_E}.$$

Now the interpolation error on Ω_E in the L_∞-norm can be estimated from (15) and the bounds for the interpolation error by using $\max_y \int_{x_i}^{x_{i+1}}|u_x|$ and $\max_x \int_{y_j}^{y_{j+1}}|u_y|$ (see [16]).

What information do we still need to prove $\varepsilon^{1/2}|u-u^I\|_1 \le CN^{-1}\ln N$? Let us consider E_1, for instance. On Ω_{11}, with $h = (4\varepsilon\ln N)N^{-1}$, we estimate

$$\|\nabla(E_1-E_1^I)\|_{0,\Omega_{11}} \le C\varepsilon N^{-1}\ln N\|\nabla^2 E_1\|_{0,\Omega_{11}}. \tag{18}$$

On $\Omega_{21}\cup\Omega_{22}$ we have

$$\begin{aligned}\|\partial_x(E_1-E_1^I)\|_{0,\Omega_{22}\cup\Omega_{21}} &\le \|\partial_x E_1\|_{0,\Omega_{21}\cup\Omega_{22}} + CN\|E_1\|_{\infty,\Omega_{21}\cup\Omega_{22}}\\ &\le C\varepsilon^{-1/2}N^{-2} + CN^{-1}.\end{aligned} \tag{19}$$

Furthermore, on Ω_{12} an anisotropic estimate is used:

$$\|\partial_x(E_1 - E_1^I)\|_{0,\Omega_{12}} \leq Ch\|\partial_{xx}E_1\|_{0,\Omega_{12}} + CN^{-1}\|\partial_{xy}E_1\|_{0,\Omega_{12}}. \tag{20}$$

Therefore, the following additional estimates are sufficient to prove (17) for the E_1-component:

$$\|(E_1)_{xx}\|_0 \leq C\varepsilon^{-3/2}, \qquad \|(E_1)_{xy}\|_0 \leq C\varepsilon^{-1/2}, \qquad \|(E_1)_{yy}\|_0 \leq C. \tag{21}$$

Now let us derive these estimates for E_1. Setting

$$E_1 = e^{-\frac{1-x}{\varepsilon}}v + w, \tag{22}$$

we define v as the solution of

$$\begin{aligned} -\varepsilon\Delta v - 2v_x + v_y &= 0 \qquad \text{in} \;\; x < 1, y > 0, \\ v|_{y=0} = 0, &\qquad v|_{x=1} = -S^*. \end{aligned} \tag{23}$$

Then v has no layers and $\|v\|_2 \leq C$. The function w satisfies

$$L_\varepsilon w = 0 \qquad \text{in} \quad 0 < x < 1, \;\; y > 0,$$

$$w|_{y=0} = 0, \qquad w|_{x=1} = 0; \qquad w|_{x=0} = -e^{-1/\varepsilon}v|_{x=0}.$$

Therefore w is exponentially small and the desired estimates for E_1 follow from the decomposition (22).

Concerning E_3, the situation is simple: in Ω_{11}, Eq. (18) is again used, while outside Ω_{11} the estimation technique of (19) works. Thus $\|E_3\|_2 \leq C\varepsilon^{-3/2}$ is sufficient, but this is a standard a priori estimate. Therefore, we have the following result:

THEOREM 2. *If $f \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, then the finite element approximation of Problem* (13) *using bilinear* (*or linear*) *elements on a Shishkin mesh satisfies*

$$\varepsilon^{1/2}|u - u^N|_1 \leq CN^{-1}\ln N.$$

4. FINITE ELEMENTS FOR THE PROBLEM WITH PARABOLIC BOUNDARY LAYERS

Now we consider the problem (1) with only the assumption that $f \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Then $u \in C^{1,\alpha}(\overline{\Omega})$ and we have the decomposition (4) with the corresponding estimates up to the first-order derivatives.

Choosing $\gamma = 1$, we define a S-mesh with the thickness $2\varepsilon \ln N$ and $2\varepsilon^{1/2}\ln N$ in the exponential and parabolic layer regions, respectively; see Fig. 2. That is, we use elements with the following order of mesh sizes in

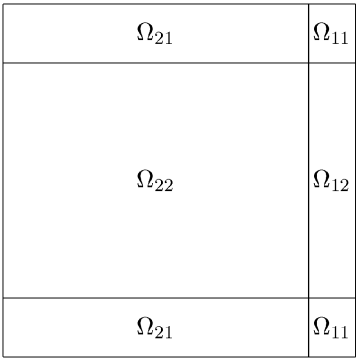

FIG. 2. S-mesh for a problem with characteristic and exponential layers.

the x- and y-directions:

$$\Omega_{22}\colon (N^{-1}, N^{-1})$$
$$\Omega_{11}\colon (h_1, h_2)$$
$$\Omega_{12}\colon (h_1, N^{-1}), \qquad \Omega_{21}\colon (N^{-1}, h_2)$$

(here $h_1 = O(\varepsilon(\ln N)N^{-1})$, $h_2 = O(\varepsilon^{1/2}(\ln N)N^{-1})$).

For estimating the discretization error, we follow the approach of Section 3, and the main difficulty lies in bounding

$$\int_\Omega (u - u^I)(u^N - u^I)_x.$$

Again we use on Ω_{22} the interpolation error estimate

$$\|u - u^I\|_{0,\Omega_{22}} \le CN^{-2} \tag{24}$$

and an inverse estimate. But the third estimate of (17) does not hold true now because

$$\text{meas } \Omega_E = O(\varepsilon^{1/2}) \quad \text{instead of} \quad O(\varepsilon) \quad \text{in Section 3.}$$

Thus on Ω_{21} we use also the L_2-interpolation error in combination with an anisotropic inverse estimate, while on $\Omega_E^* = \Omega_{12} \cup \Omega_{11}$ we can again use

$$\|u - u^I\|_{0,\Omega_E^*} \le C\varepsilon^{1/2}(N^{-1}\ln N).$$

To prove the necessary interpolation error estimates we again need estimates in Sobolev norms for the second-order derivatives. For instance, as well as the bounds

$$\|(E_2)_{xx}\|_0 \le C, \quad \|(E_2)_{xy}\|_0 \le C\varepsilon^{-1/4}, \quad \text{and} \quad \|(E_2)_{yy}\|_0 \le C\varepsilon^{-1/2}, \tag{25}$$

the estimates

$$\|(E_1)_{xx}\|_0 \le C\varepsilon^{-3/2}, \quad \|(E_1)_{xy}\|_0 \le C\varepsilon^{-1/2}, \quad \text{and} \quad \|(E_1)_{yy}\|_0 \le C. \tag{26}$$

can be proved with arguments similar to those in the proof of (21) in Section 3.

Therefore, we get finally:

THEOREM 3. *If $f \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, then the finite-element approximation of Problem* (1) *using bilinear* (*or linear*) *elements on a Shishkin mesh satisfies*

$$\varepsilon^{1/2}|u - u^N|_1 \le CN^{-1}\ln N.$$

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